

# Compatibility in Multiparameter Quantum Metrology

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Simultaneous estimation of multiple parameters in quantum metrological models is complicated by factors relating to the (i) existence of a single probe state allowing for optimal sensitivity for all parameters of interest, (ii) existence of a single measurement optimally extracting information from the probe state on all the parameters, and (iii) statistical independence of the estimated parameters. We consider the situation when these concerns present no obstacle and for *every* estimated parameter the variance obtained in the multiparameter scheme is equal to that of an optimal scheme for that parameter alone, assuming all other parameters are perfectly known. We call such models *compatible*. In establishing a rigorous framework for investigating compatibility, we clarify some ambiguities and inconsistencies present in the literature and discuss several examples to highlight interesting features of unitary and non-unitary parameter estimation, as well as deriving new bounds for physical problems of interest, such as the simultaneous estimation of phase and local dephasing.

## I. INTRODUCTION

The foundations of quantum estimation theory were laid in the sixties and seventies, with the two most significant contributions from Holevo [1] and Helstrom [2]. Since then the topic has captured the attention of both the physical and mathematical communities. Most of the activity in the physical community focused on single parameter estimation with particular focus on estimating a unitary parameter, such as phase [3–6]. In recent years, however, building on existing results on multiple parameter estimation in the mathematical literature [7–9], there have been a number of theoretical and experimental papers by physicists also addressing the multiple parameter case. These include estimating multiple-parameter unitary operators [10–19], estimating both unitary and decoherence parameters [20–22], or two decoherence parameters simultaneously [23], see [24] for a short review on the topic.

Typically, when estimating multiple parameters simultaneously, there is a trade-off in how well different parameters may be estimated. When the estimation protocol is optimized from the point of view of one parameter, the precision of estimating the remaining ones deteriorates. In such cases in order to define a meaningful concept of an optimal multiparameter estimation protocol one e.g. needs to assign weights to different parameters and ask for a protocol minimizing the weighted sum of variances of different parameters.

In this paper we investigate the conditions when the above mentioned trade-off is not present and there exists a jointly optimal multiparameter estimation protocol, meaning its performance for each of the parameters matches that of a protocol optimally designed to estimate that parameter assuming all the remaining ones are perfectly known. We choose to call such protocols *compatible*.

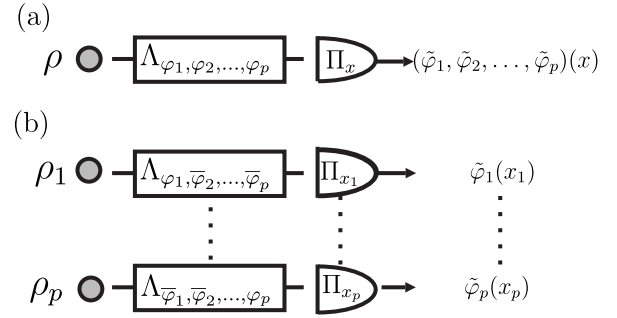


FIG. 1. (a) Simultaneous estimation of multiple parameters  $(\varphi_1, \dots, \varphi_p)$  based on results of a single measurement performed on the output of a quantum channel  $\Lambda_{\varphi_1, \dots, \varphi_p}$  acting on a single input probe  $\rho$ . (b) Separate scheme where one estimates each parameter individually using dedicated probe states and measurements, treating in every run the remaining parameters as perfectly known. We say that the parameters  $(\varphi_1, \dots, \varphi_p)$  of the quantum channel estimation model are compatible if there exists a simultaneous estimation scheme where each parameter is estimated equally well as in the optimal separate scheme (in which each of the  $p$  components utilises the same resources as the entire simultaneous scheme).

*ble*, owing to the fact that a particularly quantum feature of this trade-off occurs in the measurement stage, where it is possible that the optimal measurements for different parameters correspond to incompatible (non-commuting) observables. However, measurement compatibility is but one of several conditions we require for metrological compatibility in general.

The paper is organized as follows. In Section II we formulate the framework of multiparameter quantum metrology and discuss the requirements for compatible multiparameter estimation. We also discuss variants of the protocols depending on the use of entanglement at the input as well as at the measurement stages. In Section III we review the multiparameter classical Cramér-Rao (CR) bound, as well as two of its quantum generalizations: the quantum Fisher information (QFI) CR

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Bound and the Holevo CR bound. In Section IV we provide a simple proof for a necessary and sufficient condition for the equivalence of the QFI CR bound with the Holevo CR bound and hence asymptotic saturability of the QFI multiparameter CR bound. In Section V we consider a general scheme of multiparameter unitary estimation and provide an explicit structure of generating Hamiltonians that is necessary and sufficient to satisfy the compatibility requirements. In particular, we prove that when considering simultaneous estimation of angles of rotations of a spin  $j$  particle around different axes, the only non-trivial case satisfying the compatibility conditions is the  $j = 1$  case with the axes of rotation being orthogonal. In Section VI, we turn our attention to the compatible estimation of unitary and decoherence parameters, discussing some sufficient conditions for when this is possible. As an illustration, we analyze in more detail phase estimation in the presence of loss and local dephasing. While symmetric lossy interferometry is an example of a compatible estimation problem, the local dephasing case manifests incompatibility due to the lack of a single optimal probe even though all other conditions for simultaneous measurability as well as statistical independence are satisfied. Finally, in Section VII, we conclude the paper.

## II. FORMULATION OF THE PROBLEM

Let  $\Lambda_\varphi$  be a quantum channel depending on a set of parameters  $\varphi = (\varphi_1, \dots, \varphi_p)$  that we want to estimate by sending an input quantum probe  $\rho$  and measuring the output  $\rho_\varphi = \Lambda_\varphi(\rho)$  with a general measurement  $\{\Pi_x\}$ . Measurement results are distributed according to a probability distribution  $p(x|\varphi) = \text{Tr}(\rho_\varphi \Pi_x)$  and based on their values parameters are estimated using an estimator function  $\tilde{\varphi}(x) = (\tilde{\varphi}_1, \dots, \tilde{\varphi}_p)(x)$ , see Fig. 1 (a). Clearly, estimating multiple channel parameters simultaneously in a single estimation scheme is in general more challenging than estimating each of the parameters separately using dedicated schemes as in Fig. 1 (b). When estimating each parameter separately one is entitled to choose a probe state and a measurement which are optimal for enhancing the sensitivity of the scheme with respect to this particular parameter.

Still, a simultaneous metrology scheme may sometimes match the performance of the separate schemes (while using only the resources of one of them) provided the three following conditions are satisfied: (i) there is a single probe state  $\rho$  with which one can replace all input states  $\rho_i$  in the separate scheme preserving the maximal sensitivity of the output probe with respect to all the parameters, (ii) there is a single measurement  $\{\Pi_x\}$  that can replace all measurements  $\{\Pi_{x_i}\}$  in the separate scheme and yield optimal precision for each parameter, and finally (iii) under requirement of preserving optimal precision for estimating each individual parameter separately it should be possible to achieve independence of estimated param-

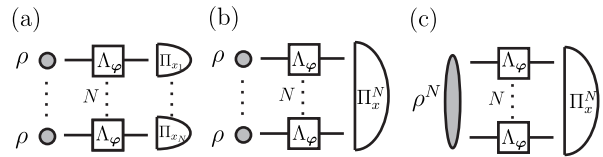


FIG. 2. Three scenarios of utilizing quantum probes in metrology: (a) “classical” scheme, where both input probes and measurements are uncorrelated, resulting in  $N$  independently and identically distributed random variables  $x_i$ , (b) collective measurement scheme, where quantum probes are uncorrelated but general collective measurements are allowed and (c) fully quantum scheme where input probes may be arbitrarily entangled and collective measurements are allowed.

eters, in the sense of vanishing off-diagonal elements of the covariance matrix, so that imperfect knowledge of one of them does not deteriorate the precision of estimating the others. If these three conditions are satisfied, the optimal scheme for any of the parameters individually is no more powerful than the scheme in which they are all estimated together. In this case we say that the channel parameters to be estimated are *compatible*.

In the above discussion we have not specified whether the channel considered acts on a single quantum probe or represents  $N$  channels acting on  $N$  possibly entangled quantum systems and yielding the output state  $\rho_\varphi^N = \Lambda_\varphi^{\otimes N}(\rho^N)$ . This last model is commonly considered in quantum metrology when investigating the problem of scaling of estimation precision with increasing probe number. In this case, one can discriminate between three scenarios that differ with respect to the extent to which quantum correlations are employed, see Fig. 2. As will be discussed in more detail further on in the paper, the question of compatibility may be strongly affected by the type of scenario we consider. Hence it is always important to clearly state which scenario one has in mind when discussing multiparameter estimation schemes. We may also combine these pictures, as is often done in practical applications, where e.g. a fully quantum scheme involving  $N$  particles is repeated  $\nu$  times in order to gather sufficient statistics. One may then meaningfully compare such schemes with the “classical ones” involving  $n = \nu N$  probes in order to study the gain arising from entangling the quantum probes and utilizing collective measurements.

## III. MULTIPARAMETER CRAMÉR-RAO BOUNDS

In this section we review the main tools of multiparameter quantum metrology based on variants of CR bounds that are used further on in this paper. In particular we stress the difference between single and multiparameter cases as well discussing reasons why metrological incompatibility may appear in different settings.

### A. Classical Multiparameter Cramér-Rao bound

First we shall consider a classical multiparameter estimation scheme. The central objects here are probability distributions  $p(\mathbf{x}|\varphi)$  of data  $\mathbf{x}$  dependent upon the parameters. This can be thought of as a quantum estimation problem where we've fixed a measurement  $\{\Pi_{\mathbf{x}}\}$  and state, thereby obtaining  $p(\mathbf{x}|\varphi) = \text{Tr} \rho_{\varphi} \Pi_{\mathbf{x}}$ . We can define the Fisher information (FI) matrix for  $m$  parameters as the  $m \times m$  matrix with entries given by

$$F_{ij}(\varphi) = \sum_{\mathbf{x}} p(\mathbf{x}|\varphi) \left( \frac{\partial \ln p(\mathbf{x}|\varphi)}{\partial \varphi_i} \right) \left( \frac{\partial \ln p(\mathbf{x}|\varphi)}{\partial \varphi_j} \right). \quad (1)$$

Crucially, this matrix allows us to define the multiparameter CR bound:

$$\text{Cov}(\tilde{\varphi}) \geq F^{-1}(\varphi), \quad (2)$$

where  $\text{Cov}(\tilde{\varphi})$  refers to the covariance matrix for a locally unbiased estimator  $\tilde{\varphi}(\mathbf{x})$ ,  $\text{Cov}(\tilde{\varphi})_{ij} = \langle (\tilde{\varphi}_i - \varphi_i)(\tilde{\varphi}_j - \varphi_j) \rangle$  and  $\langle \cdot \rangle$  represents the average with respect to the probability distribution  $p(\mathbf{x}|\varphi)$ . The above inequality should be understood as a matrix inequality. In general, we can write  $\text{Tr}[G \text{Cov}(\tilde{\varphi})] \geq \text{Tr}(GF^{-1}(\varphi))$  where  $G$  is some positive cost matrix, which allows us to asymmetrically prioritise the uncertainty cost of different parameters. As in the single parameter case, the bound is saturable in the limit of an infinite number of repetitions of an experiment using the maximum likelihood estimator [25].

The first substantial difference of multiparameter metrology from the single parameter case can already be discussed at the classical level. Assuming we've already chosen a probe state and a measurement, it may happen that the resulting FI matrix is non-diagonal. This means that the estimators for the parameters will not be independent. Considering now the separate scheme of Fig. 1b and assuming all parameters except the  $i$ -th one are perfectly known, the single parameter CR bound implies that the uncertainty of estimating the  $i$ -th parameter is lower bounded by  $\text{Var}(\tilde{\varphi}) \geq 1/F_{ii}$ . On the other hand in the simultaneous scenario of Fig. 1b according to (2) we have  $\text{Var}(\tilde{\varphi}) \geq (F^{-1})_{ii}$ . From basic algebra of positive-definite matrices, we have that  $(F^{-1})_{ii} \geq 1/F_{ii}$ , with equality holding only in the case when all off-diagonal elements  $F_{ij} = 0$ ,  $j \neq i$ . Since asymptotically the CR bound is saturable, it implies that the equivalence between the simultaneous and separate scheme in the limit of a large number of experiment repetitions can only hold if  $F$  is a diagonal matrix, and hence there are no statistical correlations between the estimators [26]. Otherwise condition (iii) for parameter compatibility is violated.

Clearly, for any real positive definite matrix one can perform an orthogonal rotation to a new basis in which the matrix is diagonal. This simply means that there are always linear combinations of the parameters for which the diagonality conditions hold. Often, however, the choice of the parameters we are interested in arise as a result of physical considerations and in this sense there

is a preferred basis in which the question of parameter compatibility has clear physical implications.

### B. Quantum Fisher Information Cramér-Rao bound

While the fundamental objects for calculating the classical FI are probability distributions of the data conditioned on the parameters to be estimated, the fundamental objects in the quantum problem are the density matrices  $\rho_{\varphi}$  dependent on these parameters. Note that here we assume that a probe state has already been selected and subjected to evolution and hence for the time being we ignore the issue of optimization over input probes.

In the quantum scenario we therefore face an additional challenge of determining the optimal measurement for extracting most of the information on the parameters of interest from the quantum states. In the single parameter case the situation is relatively simple. Maximization of the classical FI over all quantum measurements yields the quantity referred to as the QFI which can be calculated using the following formula:

$$F_Q(\varphi) = \text{tr}(\rho_{\varphi} L^2), \quad (3)$$

where  $L$  is a Hermitian matrix, the so-called symmetric logarithmic derivative (SLD), defined implicitly by  $\frac{1}{2}(L\rho_{\varphi} + \rho_{\varphi}L) = \partial_{\varphi}\rho_{\varphi}$ , where for simplicity of notation we do not explicitly write the dependence of  $L$  on  $\varphi$ . Moreover, one can always choose the projective measurement in the eigenbasis of the SLD which yields FI equal to the QFI. Hence, the QFI determines the ultimate achievable precision of estimating the parameter on density matrices  $\rho_{\varphi}$  in the asymptotic limit of an infinite number of experiment repetitions. Moreover, the fact that the QFI is additive on tensor product density matrices, in particular  $F_Q(\rho_{\varphi}^{\otimes N}) = NF_Q(\rho_{\varphi})$ , and achievable via individual measurements, implies that there is no asymptotic gain in performing collective measurements over individual ones, hence scenarios (a) and (b) in Fig. (2) are equivalent in the single parameter estimation case.

We now move on to a multiparameter scenario. A direct generalization of single parameter CR bound leads to the multiparameter QFI CR bound [1, 2] that reads:

$$\text{Cov}(\tilde{\varphi}) \geq F_Q(\varphi)^{-1}, \quad F_{Q_{ij}}(\varphi) = \frac{1}{2}\text{tr}(\rho_{\varphi}\{L_i, L_j\}), \quad (4)$$

where the braces refer to the anticommutator, whereas  $L_i$  is the SLD related to parameter  $i$ , defined analogously to the single parameter case as  $\frac{1}{2}(L_i\rho_{\varphi} + \rho_{\varphi}L_i) = \partial_{\varphi_i}\rho_{\varphi}$ . As a result, given any cost matrix  $G$ , the estimation cost is bounded by,

$$\text{Tr}[G \cdot \text{Cov}(\tilde{\varphi})] \geq \text{Tr}(GF_Q^{-1}). \quad (5)$$

Unlike in the single parameter case the above bound is not always saturable. The intuitive reason for this is incompatibility of the optimal measurements for different

parameters. Under what conditions may we nevertheless hope to saturate the bound? Given that the optimal measurement for a given parameter is formed from projectors corresponding to the eigenbasis of the SLD, we may immediately identify that if  $[L_i, L_j] = 0$  then there is a single eigenbasis for both SLDs and thus a common measurement optimal from the point of view of extracting information on  $\varphi_i$  as well as  $\varphi_j$ . However, this is only a sufficient but not a necessary condition. We discuss a necessary and sufficient condition in Sec. IV, but in preparation for this, we need to introduce a more powerful version of the multiparameter CR bound.

### C. Holevo Cramér-Rao Bound

The problem with saturability of the multiparameter QFI CR bound was realized early in the development of quantum estimation theory by Holevo [1]. He proposed a stronger multiparameter bound which we refer to as the Holevo CR bound. Its original formulation is not very explicit and therefore we prefer to use its equivalent formulation put forward in [27]. Given a cost matrix  $G$  the achievable estimation uncertainty is lower bounded by

$$\text{Tr}[G \cdot \text{Cov}(\tilde{\varphi})] \geq \min_{\{X_i\}} \{ \text{Tr}(G \cdot \text{Re}V) + \|G \cdot \text{Im}V\|_1 \}, \quad (6)$$

where  $\|\cdot\|_1$  is the operator trace norm,  $V_{ij} = \text{Tr}(X_i X_j \rho_\varphi)$ , and the minimization is performed over Hermitian matrices  $X_i$ , satisfying  $\frac{1}{2} \text{Tr}(\{X_i, L_j\} \rho_\varphi) = \delta_{ij}$ , where  $L_i$  are SLDs as defined before. The last constraint plays the role of the local unbiasedness condition.

This bound is indeed stronger than the QFI CR bound which may be appreciated by rewriting the r.h.s. of the QFI bound, Eq. (5), in the following form [27]:

$$\text{Tr}(GF_Q^{-1}) = \min_{\{X_i\}} \text{Tr}(G \cdot \text{Re}V), \quad (7)$$

with the same constraints on the  $X_i$  matrices as in the definition of the Holevo CR bound. Clearly, since the second term in Eq. (6) is positive, it implies that the QFI bound is in general weaker. As the above formula for the QFI CR bound is not widely recognized, for the sake of completeness and anticipating further discussion of the saturability issue, we provide a proof of it below.

Let us write the solution to the minimization problem of the r.h.s of Eq. (7) explicitly using the Lagrange multiplier method. Introducing Lagrange multipliers  $\lambda_{ij}$  we need to minimize

$$\frac{1}{2} \sum_{ij} G_{ij} \text{Tr}(\rho_\varphi \{X_i, X_j\}) - \lambda_{ij} [\delta_{ij} - \frac{1}{2} \text{Tr}(\rho_\varphi \{X_i, L_j\})] \quad (8)$$

over Hermitian  $X_i$ . Each  $n$ -dimensional Hermitian matrix  $X_i$  may be parametrized by  $n^2$  real parameters. Taking the derivatives over each of these produces a set of

matrix equations,

$$\forall_i \sum_j G_{ij} \{ \rho_\varphi, X_j \} - \lambda_{ij} \{ \rho_\varphi, L_j \} = 0. \quad (9)$$

Taking

$$X_i = \sum_j (G^{-1} \Lambda)_{ij} L_j, \quad (10)$$

where by  $\Lambda$  we denote the matrix of Lagrange multipliers  $(\Lambda)_{ij} = \lambda_{ij}$  it is clear that Eq. (9) is satisfied. Moreover, the constraint condition  $\frac{1}{2} \text{Tr}(\{X_i, L_j\} \rho_\varphi) = \delta_{ij}$  reads:

$$\frac{1}{2} (G^{-1} \Lambda)_{ik} \text{Tr}(\{L_k, L_j\} \rho_\varphi) = \delta_{ij}. \quad (11)$$

This implies that the Lagrange multiplier matrix must be chosen so that:

$$G^{-1} \Lambda F_Q = \mathbb{1}. \quad (12)$$

As a result the solution to the minimization problem reads

$$X_i = \sum_j (F_Q^{-1})_{ij} L_j \quad (13)$$

and utilizing the fact that QFI matrix is symmetric we get

$$\text{Tr}(G \cdot \text{Re}V) = \text{Tr}(GF_Q^{-1} F_Q F_Q^{-1}) = \text{Tr}(GF_Q^{-1}), \quad (14)$$

which ends the proof.

Even though the Holevo CR bound is tighter than the QFI one, it is still not always saturable with separable measurements. However, it is saturable for Gaussian state shift models where the parameters are encoded in the first moment displacements [1]. Even more interestingly, thanks to the theory of quantum local asymptotic normality (QLAN) [28–30] which asymptotically maps any quantum estimation problem performed on a large number of copies of a quantum state to a corresponding Gaussian shift model, the Holevo CR bound is asymptotically achievable in this case as well. Since the mapping does not respect separation into single copy subsystems, collective measurement may in general be required to saturate the Holevo CR bound. Hence, for all schemes depicted in Fig. 2b the Holevo CR bound provides an ultimate asymptotically saturable multiparameter CR bound.

## IV. MULTIPARAMETER COMPATIBILITY

### A. Saturability of multiparameter Quantum Fisher Information Cramér-Rao bound

As we mentioned before if the SLDs  $L_i$  corresponding to the different parameters commute, there is no additional difficulty in extracting optimal information from



a state on all parameters simultaneously. If they do not commute, however, this does not immediately imply that it is impossible to simultaneously extract information on all parameters with precision matching that of the separate scenario for each.

A weaker condition has appeared in a number of papers [7, 13, 21, 22] which states that the multiparameter QFI CR bound can be saturated provided

$$\text{Tr}(\rho_\varphi [L_i, L_j]) = 0, \quad (15)$$

where not the commutator itself but only its expectation value on the probe state is required to vanish. Henceforth we shall refer to this as the *commutation condition*. This condition was first identified as necessary and sufficient by Matsumoto [7] for the case when  $\rho_\varphi$  is a pure state, upon which the criterion is equivalent to the existence of *some* pair of SLDs which commute, given that SLDs are not unique on pure states. It is then possible to find an optimal measurement as the common eigenbasis of these SLDs. This implies that for unitary evolution on pure states, satisfaction of the commutation condition coincides with the existence of commuting Hamiltonians which *could* have generated the evolution on the given probe.

For mixed states, this condition has met some small inconsistencies in its usage, being variously identified as sufficient [13] or necessary and sufficient [22] in different papers. To clear up this confusion we present a derivation of this criterion, which to the best of our knowledge has not been provided before in such a simple and direct manner.

First of all, we consider a scenario where estimation is performed on multiple independent copies of the output state  $\rho_\varphi$  and allow for collective measurements as in Fig. 2 (b). Without such assumptions not much can be said in general, since even at the single-parameter level, the CR bound is in general not saturable without invoking the asymptotic limit of many independent repetitions as in Fig. 2 (a).

We know already from the discussion in Sec. III C that in this case the Holevo CR bound is asymptotically achievable thanks to QLAN theory. Hence, to prove asymptotic saturability of the multiparameter QFI CR bound it is enough to prove that it is equivalent to the Holevo CR bound if and only if the commutation condition (15) holds.

*Proof* For the sake of the proof we assume that both the cost matrix  $G$  and QFI matrix  $F_Q$  are strictly positive. These are natural assumptions since otherwise if some eigenvalues of  $G$  were zero, uncertainty in some parameter combinations would not be penalized whereas if some eigenvalues of  $F_Q$  were zero, it would be impossible to estimate some of the parameters with finite precision.

Let us first prove sufficiency of (15) and assume that  $\text{Tr}([L_i, L_j]\rho_\varphi) = 0$ . We have seen that when calculating the minimum in the formula for the QFI bound using (7) we have found that the optimal  $X_i = \sum_j (F_Q^{-1})_{ij} L_j$  are linear combinations of  $L_i$ . Since  $\text{Tr}([L_i, L_j]\rho_\varphi) = 0$  for all

$i, j$  it implies that the the same holds for all their linear combinations and hence  $\text{Tr}([X_i, X_j]\rho_\varphi) = 0$  for all  $i, j$ . This, however, implies that the same set of  $X_i$  minimizes the formula for the Holevo bound as it makes the second term in (6) equal to zero.

To prove the necessity we assume that the Holevo bound coincides with the QFI bound and hence for the  $X_i$  that minimize both (6) and (7) the second term in (6) must be equal to zero. Since  $G$  is strictly positive, this implies that the matrix  $\text{Im}V$  must be zero and hence  $\text{Tr}([X_i, X_j]\rho_\varphi) = 0$  for all  $i, j$ . On the other hand, we know that the  $X_i$  minimizing (7) have the form  $X_i = \sum_j (F_Q^{-1})_{ij} L_j$ . Inverting this formula we get  $L_i = \sum_j (F_Q)_{ij} X_j$  and hence  $\text{Tr}([L_i, L_j]\rho_\varphi) = 0$  for all  $i, j$  ■.

It's worth stressing the different implications of the commutation condition on pure states and on mixed states. In the case of pure states, as already mentioned above, the commutation relation implies that there is an individual measurement that allows saturation of the QFI CR bound as in Fig. 2a. On the other hand, for mixed states, collective measurements on multiple copies may be necessary in general to achieve the bound as in Fig. 2b. This is due to the fact that the Holevo CR bound is guaranteed to be saturable provided one takes the asymptotic limit of many independent copies of a state, while the correspondence to Gaussian states via QLAN theory implicitly does not invoke limitations on the allowed set of measurements.

## B. Conditions for multiparameter compatibility

Combining the commutation condition with the parameter independence condition discussed in Sec. III A which requires off-diagonal QFI matrix entries to be zero, we arrive at a necessary requirement for multiparameter compatibility which reads

$$\forall_{i \neq j} \text{Tr}(L_i L_j \rho_\varphi) = 0. \quad (16)$$

Plugging in an explicit form for the SLDs

$$L_i = 2 \sum_{m,n} \frac{\langle \psi_m | (\partial_{\varphi_i} \rho_\varphi) | \psi_n \rangle}{p_m + p_n} |\psi_m\rangle \langle \psi_n|, \quad (17)$$

where  $p_{m,n}$  and  $|\psi_{m,n}\rangle$  are the eigenvalues and eigenvectors of the state  $\rho_\varphi = \sum_k p_k |\psi_k\rangle \langle \psi_k|$  from which parameters are to be estimated. The compatibility condition (16) can now be written as:

$$\forall_{i \neq j} \sum_{m,n} \frac{p_m}{(p_m + p_n)^2} \langle \psi_m | \partial_{\varphi_i} \rho_\varphi | \psi_n \rangle \langle \psi_n | \partial_{\varphi_j} \rho_\varphi | \psi_m \rangle = 0. \quad (18)$$

On top of this we must not forget the final condition which demands the existence of a single probe state that provides maximum QFIs for all the parameters.

In summary, we may decompose the demands of simultaneous estimation into several layers of stringency.

The first is the existence of a single probe state yielding maximum possible values of QFIs for all parameters of interest. Second is the requirement of the existence of compatible measurements on the output states which ensures the saturability of the QFI CR bound and the last one is the requirement that the QFI matrix is diagonal which enables independent estimation of the parameters.

If all these conditions hold, the optimal metrological strategy will not depend on the choice of the cost matrix  $G$  and the ultimate bounds on estimation precision are found in the same way as in the case of single parameter estimation.

## V. UNITARY PARAMETER ESTIMATION

Let us first treat the case of multiple unitary parameter estimation, which has been considered in a number of papers [10–19] and ask under what conditions we can have multiparameter compatibility. We consider unitary evolution acting on the input probe state to be of the form

$$U_{\varphi} = e^{i \sum_k H_k \varphi_k}. \quad (19)$$

Thanks to convexity of the QFI we can always assume the input state to be pure  $\rho = |\psi\rangle\langle\psi|$ . Since the evolution is unitary, the output state will be pure as well  $|\psi_{\varphi}\rangle = U_{\varphi}|\psi\rangle$ . For pure states the SLDs can be explicitly written as:

$$L_i = 2(|\psi_{\varphi}^{(i)}\rangle\langle\psi_{\varphi}| + |\psi_{\varphi}\rangle\langle\psi_{\varphi}^{(i)}|) \quad (20)$$

where  $|\psi_{\varphi}^{(i)}\rangle = \partial_{\varphi_i}|\psi_{\varphi}\rangle$ . For the moment, for the sake of clarity we consider estimation performed around the point where all  $\varphi_k = 0$ . In this case

$$L_i = 2i(H_i|\psi\rangle\langle\psi| - |\psi\rangle\langle\psi|H_i). \quad (21)$$

As a result, the compatibility condition (16) takes the form

$$\forall_{i \neq j} \langle\psi|(\langle H_i\rangle - H_i)(\langle H_j\rangle - H_j)|\psi\rangle = 0, \quad (22)$$

where  $\langle H_i\rangle = \langle\psi|H_i|\psi\rangle$ . Additionally, apart from fulfilling the above orthogonality conditions we must make sure that the single input probe yields optimal QFI with respect to all parameters. The QFI for the  $i$  parameter is simply proportional to the variance of  $H_i$ :

$$(F_Q)_{ii} = \langle\psi|(\langle H_i\rangle - H_i)^2|\psi\rangle \quad (23)$$

and is uniquely maximized by a probe state which is an equally weighted superposition of eigenstates  $|-\rangle_i, |+\rangle_i$  of  $H_i$  corresponding to the minimal and the maximal eigenvalues  $\lambda_i^-, \lambda_i^+$  respectively [31]

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|-\rangle_i + |+\rangle_i). \quad (24)$$

The above form of  $|\psi\rangle$  should be valid irrespective of index  $i$ . Clearly, we have freedom to adjust the relative phases in the above expression, but we can also assume that they are incorporated in the definition of the eigenstates themselves. Without losing generality, let us shift the Hamiltonians  $H_i \rightarrow H_i - \frac{\lambda_i^+ + \lambda_i^-}{2} \mathbb{1}$  so that  $\lambda_i^- = -\lambda_i^+ = -\lambda_i$  and hence  $\langle H_i\rangle = 0$  on the optimal probe state. Plugging the form of the optimal state (24) into (22) we get

$$\forall_{i \neq j} (\langle +|_i - \langle -|_i)(|+\rangle_j - |-\rangle_j) = 0. \quad (25)$$

After some basic algebra this implies that the extremal eigenvectors of  $H_i$  must necessarily be of the form:

$$|+\rangle_i = \frac{1}{\sqrt{2}}(|\psi\rangle + |\xi_i\rangle), \quad (26)$$

$$|-\rangle_i = \frac{1}{\sqrt{2}}(|\psi\rangle - |\xi_i\rangle), \quad (27)$$

where  $\langle \xi_i|\xi_j\rangle = \delta_{ij}$  and all  $|\xi_i\rangle$  are orthogonal to  $|\psi\rangle$ . The above formulas express the most general requirements on the eigenvectors of the generating Hamiltonians for compatible metrology to be achievable in this evolution model. As will be shown in the example below these are rather stringent conditions. One might object that e.g. in the case where all generators  $H_i$  are equal there should be no difficulty in estimating simultaneously multiple parameters since the optimal input probe state and the optimal measurements are identical for all  $\varphi_i$ . Note however that such a model provides us only with the information on the total accumulated phase  $\sum_i \varphi_i$  and therefore the statistical independence condition is not satisfied, and even worse, the QFI matrix is degenerate.

To end this general discussion, let us go back to the more general case of  $\varphi_i \neq 0$ . In this case all the above discussion is valid up to replacement of all  $H_i$  operators appearing in formulas from Eq. (22) onwards with  $H_i^S = U_{\varphi}^{\dagger} \mathcal{S}_i[H_i e^{i \sum_k H_k \varphi_k}]$ , where  $\mathcal{S}_i$  represents a symmetrization operation which acts when encountering any product of  $H_i$  with other operators  $H_{k \neq i}$  that do not commute with it. It performs a normalized symmetrization of this product, so e.g.  $\mathcal{S}_1[H_1 H_2^2] = \frac{1}{3}(H_1 H_2^2 + H_2 H_1 H_2 + H_2^2 H_1)$ . The above considerations may also be easily adapted to the case where different parameter unitaries act sequentially i.e.  $U_{\varphi} = \prod_k e^{i H_k \varphi_k}$ , by replacing  $H_i^S$  with  $(\prod_{k=1}^{i-1} e^{i H_k \varphi_k}) H_i (\prod_{k=i}^p e^{i H_k \varphi_k})$ .

### A. Two-parameter estimation of a spin rotation

Let us consider a spin  $j$  particle, with associated angular momentum operator  $\vec{S} = (S_x, S_y, S_z)$  and consider unitary two-parameter evolution of the form:

$$U_{\varphi_1, \varphi_2} = e^{i \varphi_1 \vec{n}_1 \cdot \vec{S} + i \varphi_2 \vec{n}_2 \cdot \vec{S}}, \quad (28)$$

where the  $H_i$  generating the unitary transformation now correspond to different directions of the spin operators

$H_i = \vec{n}_i \cdot \vec{S}$ . For simplicity we focus on estimation around  $\varphi_1 = \varphi_2 = 0$  point, though the discussion remains qualitatively equivalent when  $\varphi_i \neq 0$ . Let  $|m\rangle_{\vec{n}}$ ,  $m \in -j, \dots, j$  denote the basis constructed from eigenvectors of the  $\vec{n} \cdot \vec{S}$  operator with projection value  $m$ . According to previous discussion the optimal state needs to have the form

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|-j\rangle_{\vec{n}_1} + |+j\rangle_{\vec{n}_1}) = \frac{1}{\sqrt{2}}(|-j\rangle_{\vec{n}_2} + |+j\rangle_{\vec{n}_2}), \quad (29)$$

and clearly  $\langle H_i \rangle = 0$ . Let  $\alpha$  be the angle between directions  $\vec{n}_1$  and  $\vec{n}_2$ . Using standard theory of angular momentum we may expand states  $|\pm j\rangle_{\vec{n}_2}$  in the basis  $|m\rangle_{\vec{n}_1}$  as follows

$$|+j\rangle_{\vec{n}_2} = \sum_{m=-j}^j \binom{2j}{j+m}^{\frac{1}{2}} \sin^{j+m} \frac{\alpha}{2} \cos^{j-m} \frac{\alpha}{2} |m\rangle_{\vec{n}_1}, \quad (30)$$

$$|-j\rangle_{\vec{n}_2} = \sum_{m=-j}^j (-1)^{j-m} \binom{2j}{j+m}^{\frac{1}{2}} \cos^{j+m} \frac{\alpha}{2} \sin^{j-m} \frac{\alpha}{2} |m\rangle_{\vec{n}_1}, \quad (31)$$

where we have neglected any possible relative phases that might appear in the above decomposition as they are irrelevant in the following. Rewriting the formula for  $|+j\rangle_{\vec{n}_2}$  as

$$|+j\rangle_{\vec{n}_2} = \cos^{2j} \frac{\alpha}{2} |-j\rangle_{\vec{n}_1} + \sin^{2j} \frac{\alpha}{2} |j\rangle_{\vec{n}_1} + \sum_{m=-j+1}^{j-1} \dots \quad (32)$$

and comparing it with the compatibility conditions (26) we see that the only possibility of satisfying them is to take  $\alpha = \pi/2$  and  $j = 1$  in which case we obtain

$$|+1\rangle_{\vec{n}_2} = \frac{1}{2}(|-1\rangle_{\vec{n}_1} + |1\rangle_{\vec{n}_1}) + \frac{1}{\sqrt{2}}|0\rangle_{\vec{n}_1}, \quad (33)$$

$$|-1\rangle_{\vec{n}_2} = \frac{1}{2}(|-1\rangle_{\vec{n}_1} - |1\rangle_{\vec{n}_1}) - \frac{1}{\sqrt{2}}|0\rangle_{\vec{n}_1}, \quad (34)$$

resulting in estimation precision  $\Delta^2\varphi_1 = \Delta^2\varphi_2 = 1/4$ . With this example it is clear how restrictive the multiparameter compatibility conditions in metrology are. The fact that for spin  $j = 1/2$  there is no possibility for satisfying compatibility conditions is clear from (26) as at least three dimensional space is required to have three orthogonal states  $|\psi\rangle, |\xi_1\rangle, |\xi_2\rangle$ . It is, however, nontrivial that the only case where multiparameter compatibility can be satisfied is  $j = 1$  for rotations around two perpendicular axes. Given that we are working in the pure state case, it is always possible to find a measurement on a single spin that achieves the quantum CRB. The following projection measurement suffices,

$$\begin{aligned} \Pi_1 &= \frac{1}{2}(|+1\rangle_{\vec{n}_1} + |-1\rangle_{\vec{n}_1})(\langle+1|_{\vec{n}_1} + \langle-1|_{\vec{n}_1}), \\ \Pi_2 &= \frac{1}{2}(|+1\rangle_{\vec{n}_1} - |-1\rangle_{\vec{n}_1})(\langle+1|_{\vec{n}_1} - \langle-1|_{\vec{n}_1}), \\ \Pi_3 &= \mathbb{1} - \Pi_1 - \Pi_2. \end{aligned}$$

From the above discussion it is also clear that there is no possibility to estimate three different rotation directions in a compatible way since the only promising case  $j = 1$  corresponds to a three dimensional space whereas compatibility of three different rotation parameters require at least a four dimensional space according to (26).

The results presented above can be immediately applied to the case when  $N$  qubits experience independent two parameter rotations according to the following unitary

$$U_{\varphi_1, \varphi_2} = \left( e^{\frac{i}{2}(\varphi_1 \vec{n}_1 \cdot \vec{\sigma} + \varphi_2 \vec{n}_2 \cdot \vec{\sigma})} \right)^{\otimes N}, \quad (35)$$

as in this case the optimal input probe state lives in the fully symmetric subspace which is isomorphic to spin  $j = N/2$  space. It is therefore clear that while for a single qubit ( $N = 1$ ) undergoing simultaneous rotation around two axes, the compatibility conditions cannot be satisfied, they can be achieved when considering  $N = 2$  case and an appropriately chosen entangled input; essentially entanglement takes us from a highly incompatible case to full compatibility with Heisenberg scaling in two parameters at once (but only for  $N = 2$ ). This fact can be confirmed by inspecting results presented in [13], where the sum of variances of two angles of rotations was minimized, and noticing that only in the case of  $N = 2$ , the obtained result indeed corresponds to the optimal separate scenario. For higher dimensional  $N$  the Heisenberg bound is no longer achievable in both parameters. If we choose GHZ-type states, then we can achieve Heisenberg  $1/N^2$  scaling in one parameter, but classical  $1/N$  scaling in the other. Other states can achieve different trade-offs; for even- $N$  qubit Dicke states with  $\frac{N}{2}$  excitations (in the direction mutually orthogonal to  $\vec{n}_1$  and  $\vec{n}_2$ ), both parameters have a Fisher information of  $\frac{N^2}{2} + N$ , which asymptotically retains quadratic scaling but with a  $1/2$  prefactor.

## VI. HYBRID UNITARY + NON-UNITARY PARAMETER ESTIMATION

In the previous section we have seen that the compatibility conditions in the case of multiple unitary parameters are very demanding and can be satisfied only in very special situations. In this section we focus on the case when one of the parameters  $\varphi$  is unitary whereas the other one, which we denote by  $\eta$  enters via a non-unitary part of the evolution as e.g. a decoherence strength parameter.

This scenario has been considered before in several models such as the estimation of loss and phase in an interferometer [22], as well as the estimation of phase with collective [20] and independent [21] dephasing. Here we want to investigate the possibility of satisfying the compatibility conditions in such situations.

Before considering specific schemes, let us first identify some general sufficient criteria for the compatibility con-

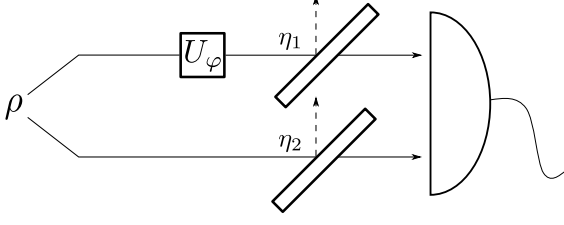


FIG. 3. A schematic of a general lossy interferometer with input state  $\rho$ . We model the losses by a beam-splitter. In [22], a scheme was considered with transmissivity  $\eta_2 = 1$ , leading to one arm containing both the loss and phase parameters. We balance the interferometer by choosing  $\eta_1 = \eta_2 = \eta$ .

dition as expressed by Eq. (16) and ignore for the moment the requirement for the existence of common optimal input probe state. The explicit form of the compatibility condition (18) can be written as:

$$\sum_{m,n} \frac{p_m}{(p_m + p_n)^2} \langle \psi_m | \partial_\varphi \rho_{\varphi\eta} | \psi_n \rangle \langle \psi_n | \partial_\eta \rho_{\varphi\eta} | \psi_m \rangle = 0, \quad (36)$$

where  $p_n$ ,  $|\psi_n\rangle$  are eigenvalues and eigenvectors of  $\rho_{\varphi\eta}$ .

Let us assume that the decoherence parameter  $\eta$  induces a “classical” evolution in the sense that

$$\partial_\eta \rho_{\varphi\eta} = \sum_k (\partial_\eta p_k) |\psi_k\rangle \langle \psi_k| \quad (37)$$

so that only the eigenvalues of the density matrix depend on the parameter and the state remains diagonal in its initial eigenbasis. This makes all off-diagonal terms  $m \neq n$  in (36) zero. However, since the second parameter is unitary,

$$\langle \psi_n | \partial_\varphi \rho_{\varphi\eta} | \psi_n \rangle = p_n \partial_\varphi \langle \psi_n | \psi_n \rangle = 0 \quad (38)$$

and hence the diagonal terms are zero as well, guaranteeing the compatibility condition to hold.

There are more involved cases when the decoherence parameter  $\eta$  influences not only the eigenvalues but the form of the eigenvectors of  $\rho_{\varphi,\eta}$  as well. It might happen that even though individual terms in (36) are non-zero they sum up to zero in the end. Such situations need to be dealt with on a case by case basis.

#### A. Estimation of phase and loss in an interferometer.

Consider an interferometer with equal loss in *both* arms, as presented in Fig. 3, where the goal is to estimate both the relative phase delay  $\varphi$  between the arms as well as the transmission coefficient  $\eta$ . We choose for our input states to be fixed photon-number states, for

which a general bipartite state is given by

$$|\psi\rangle = \sum_{k=0}^N \alpha_k |k, N-k\rangle. \quad (39)$$

After passing through the interferometer the resultant state is

$$|\psi_{\varphi\eta}\rangle = \sum_{k=0}^N \sum_{l_2=0}^{N-k} \sum_{l_1=0}^k \alpha_k e^{ik\varphi} \sqrt{B_{l_1 l_2}^k} |k, N-k\rangle \otimes |l_1, l_2\rangle, \quad (40)$$

where the additional two modes represent photons lost from respectively the upper and the lower arm and

$$B_{l_1 l_2}^k = \binom{k}{l_1} \binom{N-k}{l_2} \eta^{N-l_1-l_2} (1-\eta)^{l_1+l_2}. \quad (41)$$

On tracing out the auxiliary modes, we obtain a density matrix

$$\rho_{\varphi\eta} = \bigoplus_l \sum_{l_1} |\psi_{l_1, l-l_1}\rangle \langle \psi_{l_1, l-l_1}|, \quad (42)$$

where different  $l = l_1 + l_2$  sectors represent different total number of photons lost while

$$|\psi_{l_1 l_2}\rangle = \sum_{k=l_1}^{N-l_2} \alpha_k e^{ik\varphi} \sqrt{B_{l_1 l_2}^k} |k-l_1, N-k-l_2\rangle \quad (43)$$

are subnormalized states corresponding to the situation of losing  $l_1$  and  $l_2$  photons in the upper and the lower arm respectively. Note that states  $|\psi_{l_1, l-l_1}\rangle$  living in a single  $l$  sector are in general not orthogonal and hence should not be understood as eigenvectors of  $\rho_{\varphi\eta}$ . Still, owing to the fact that

$$\partial_\eta B_{l_1 l_2}^k = c_{N,l} B_{l_1 l_2}^k, \quad c_{N,l} = \frac{N-l}{\eta} - \frac{l}{1-\eta} \quad (44)$$

we eventually arrive at:

$$\partial_\eta \rho_{\varphi\eta} = \bigoplus_l c_{N,l} \sum_{l_1} |\psi_{l_1, l-l_1}\rangle \langle \psi_{l_1, l-l_1}|, \quad (45)$$

implying that upon differentiation the whole block corresponding to a fixed  $l$  is multiplied by the same constant factor. This means that only the eigenvalues of the density matrix are changed with variations of the parameter  $\eta$  and hence we conclude that variations of  $\eta$  induce the “classical” evolution. From the general considerations presented in the beginning of this section this implies that the compatibility criterion is satisfied.

More specifically, a brief calculation shows that the SLD  $L_\eta$  decomposes into a weighted sum of projectors onto the blocks of the density matrix and thus an optimal measurement for loss is simply the set of projectors onto each block of constant  $l$ . The resultant Fisher information reads  $\sum_L c_{N,l}^2 P(l|\eta)$ , where most importantly  $P(l|\eta) = \text{Tr} \rho_{\varphi\eta} \Pi_l = \binom{N}{l} \eta^{N-l} (1-\eta)^l$  does not depend on



the input state. As a result we simply get a binomial distribution of total numbers of photons lost, and sampling this is the most informative thing we can do to learn  $\eta$ . The corresponding QFI reads  $(F_Q)_{\eta\eta} = \frac{N}{\eta(1-\eta)}$ .

Since  $\partial_\varphi \rho_{\varphi\eta}$  does not mix blocks of different total photon-number (as phase shifts do not alter photon number), we find that  $L_\varphi$  can be decomposed into the same blocks as  $L_\eta$ , and since  $L_\eta$  simply acts as a multiple of the identity block-wise, they properly commute, not just under expectation value. Hence no collective measurements on multiple copies of the quantum state are necessary to saturate the QFI CR bound, even though we are in the mixed state case.

Finally, we do not face the problem of determining a common optimal input probe. Since precision of estimating  $\eta$  is state independent we simply take the optimal state maximizing QFI for phase estimation [32–34]. Taking the asymptotic analytical formula for optimal QFI in the limit of large  $N$  [34–37] and assuming  $\eta < 1$  we summarize this section by providing the achievable precision of compatible simultaneous phase and loss estimation:  $\Delta^2\varphi = \frac{1-\eta}{\eta N}$ ,  $\Delta^2\eta = \frac{\eta(1-\eta)}{N}$ .

## B. Estimation of phase and dephasing

Let us now consider  $N$  qubits undergoing evolution composed of unitary phase combined with individual dephasing processes. Each qubit is affected independently and the output  $N$ -qubit density matrix reads:

$$\rho_{\varphi\eta} = \Lambda_{\varphi\eta}^{\otimes N}(\rho), \quad (46)$$

where

$$\Lambda_{\varphi\eta}(X) = U_\varphi \left( \sum_{i=0}^1 K_i X K_i^\dagger \right) U_\varphi^\dagger, \quad (47)$$

$U_\varphi = \exp(i\varphi\sigma_z/2)$ , while the two Kraus operators read  $K_0 = \sqrt{\frac{1+\eta}{2}}\mathbb{1}$  and  $K_1 = \sqrt{\frac{1-\eta}{2}}\sigma_z$ .

In the case of  $N = 1$ , any state on the equator of the Bloch sphere is known to be optimal both from the point of view of estimating phase as well as the dephasing coefficient [38]. Taking  $\rho = |+\rangle\langle+|$ , with  $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$  we find the output state

$$\rho_{\varphi\eta} = \eta|\varphi\rangle\langle\varphi| + (1-\eta)\mathbb{1}/2, \quad (48)$$

where  $|\varphi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + e^{i\varphi}|1\rangle)$ . This is clearly the case where  $\eta$  induces “classical” evolution, changing the eigenvalues without changing the eigenvectors and hence the compatibility condition is immediately satisfied.

Still, as discussed in detail in [21], saturating the QFI CR bound in this case requires application of collective measurement on multiple copies of the state, unlike in the example of estimating phase and loss. Discussion in [21] was restricted to probes being products of single qubit

states. Here we want to investigate the problem of simultaneous estimation in case of arbitrary entangled input states of  $N$  qubits, since utilizing entangled input probes is indispensable to reach the optimal phase estimation performance in the presence of dephasing [36, 39, 40]. It is known that the optimal input states are highly symmetric, exhibiting both permutational symmetry of the qubits, and also a parity symmetry under bit flips, i.e. they are invariant under  $\sigma_x^{\otimes N}$  where  $N$  refers to the number of qubits. We will thus investigate the class of  $N$ -qubit states defined by these two kinds of symmetries. This assumption is further justified by the fact the states optimized from the point of view of estimating the dephasing coefficient satisfy these symmetries as they are simply the product states  $|+\rangle^{\otimes N}$  [41, 42] yielding the optimal estimation precision  $\Delta^2\eta = \frac{1-\eta^2}{N}$ . Let us also note here, that in the limit of large  $N$ , simple classes of one- and two-axis spin-squeezed states reaches the optimal phase estimation precision limit given by  $\Delta^2\varphi = \frac{1-\eta^2}{\eta^2 N}$  [36, 40].

Due to the high degree of symmetry, it is convenient to shift to angular momentum notation. In general, we write  $|j, m\rangle$  to denote a general angular momentum eigenstate where for  $N$  qubits  $0 \leq j \leq \frac{N}{2}$  and  $j$  goes between these limits in integer steps (with a lower bound of  $\frac{1}{2}$  for odd  $N$ ) and similarly  $-j \leq m \leq j$ , where  $m$  also increases in integer steps. We can then write the permutationally symmetric pure input states as  $|\psi\rangle = \sum_m \alpha_m |\frac{N}{2}, m\rangle$ , where  $\sum_m |\alpha_m|^2 = 1$ .

After experiencing local dephasing the state will no longer be supported on the fully symmetric subspace  $j = N/2$  but will preserve permutational invariance on the level of the density matrix. A particularly useful construction for the decomposition of the output state of this evolution is found in [43]

$$\rho_{\varphi\eta} = \sum_j \sum_{m, m'} h(N, j, m, m', \eta) e^{i\varphi(m' - m)} |j, m\rangle\langle j, m'|, \quad (49)$$

where the actual form of  $h(N, j, m, m', \eta)$  coefficients is quite involved and we refer the interested reader to [43] as it has no relevance for further discussion here. In the above expression it is implicitly assumed that the state has the same form on all multiplicity subspaces corresponding to the same  $j$  and we write the state using a simplified notation as if there were no multiplicity of spaces with given  $j$ .

Let us consider state  $\rho_\eta$ , which is an output state *before* implementing the phase evolution—this is permitted because the actions of phase and dephasing commute. We first discuss a further simplification on the structure of  $\rho_\eta$ . The parity symmetry implies that within each of the blocks of constant  $j$ , there exists a further splitting according to the irreducible representations of the parity operator. The parity operator only has one-dimensional irreducible representations corresponding to the trivial and to the alternating representation. The eigenvectors of  $\rho_\eta$  can then be chosen to have either even or odd parity.

Given the block diagonal structure, the  $i^{\text{th}}$  even parity vector in the  $j$  subspace can be expressed as  $|\psi'_{\text{even},i}\rangle = \sum_m e_{i,m}^j(|j,m\rangle + |j,-m\rangle)/\sqrt{2}$ , where  $\sum_m |e_{i,m}^j|^2 = 1$ . Similarly, all odd parity eigenvectors have the structure  $|\psi'_{\text{odd},i}\rangle = \sum_m o_{i,m}^j(|j,m\rangle - |j,-m\rangle)/\sqrt{2}$ , where  $\sum_m |o_{i,m}^j|^2 = 1$ .

Now consider the decomposition of the density matrix in terms of such eigenvectors  $\rho_\eta = \sum_i p_i |\psi'_i\rangle\langle\psi'_i|$ . The unitary phase only serves to alter the eigenstates. Thus, after the phase unitary the density matrix is  $\rho_{\eta\varphi} = \sum_i p_i |\psi_i\rangle\langle\psi_i|$ , where  $|\psi_i\rangle = U(\varphi)|\psi'_i\rangle$ . Due to this  $\langle\psi_i|\partial_\eta\psi_k\rangle = \langle\psi_i|U^\dagger(\varphi)\partial_\eta(U(\varphi)|\psi_k\rangle) = \langle\psi'_i|\partial_\eta\psi'_k\rangle$ . This simplifies the calculation of  $\langle\psi_i|\partial_\eta\rho_{\eta\varphi}|\psi_k\rangle$  terms when  $i \neq k$ . Most significantly, one can observe that for  $\psi'_i$  and  $\psi'_k$  from subspaces corresponding to different parities,  $\langle\psi_i|\partial_\eta\psi_k\rangle = 0$ .

This is because the subspaces as a whole do not change with  $\eta$ . Considering the almost-trivial example of the decomposition of two qubits into triplets and singlets, the singlet space always remains completely separate from the triplet space and it will not overlap with any combination of triplets regardless of  $\eta$ . The parity subspaces behave similarly.

This eliminates approximately half of the terms of equation (18). We turn our focus to the remaining terms which include  $|\psi_i\rangle$  and  $|\psi_j\rangle$  from the *same* parity subspace. We will treat the even parity case, but the proof for odd parity is identical. After the phase unitary, the eigenstates become  $|\psi_{\text{even},i}\rangle = \sum_m e_{i,m}^j(e^{-i\frac{\varphi}{2}m}|j,m\rangle + e^{i\frac{\varphi}{2}m}|j,-m\rangle)$ . Differentiating this state with respect to  $\varphi$  induces a sign difference between the two terms sharing the coefficient  $e_{i,m}^j$ . Using the orthonormality of  $|j,m\rangle$  gives us  $\langle\psi_{\text{even},i}|\partial_\varphi\psi_{\text{even},k}\rangle = \sum_m e_{i,m}^{j,*}e_{k,m}^j(m-m) = 0$ . Thus every numerator term,  $\langle\psi_i|\partial_\eta\rho|\psi_j\rangle\langle\psi_j|\partial_\varphi\rho|\psi_i\rangle$ , of  $\text{Tr}(\rho_{\eta,\varphi}L_\varphi L_\eta)$  is equal to 0 and we can simultaneously estimate the parameters.

There still remains the issue of existence of a common input state optimal both for  $\varphi$  and  $\eta$  simultaneously. This fact is obvious for the single qubit case,  $N = 1$ , as any equatorial qubit state is an optimal probe from the point of view of both parameters. For  $N \geq 2$ , however, this is no longer true. We have performed a numerical search which showed that when optimizing probe states from the point of view of estimating two-parameters simultaneously we face a trade-off and the optimal state for joint estimation depends on the weighting of importance between dephasing and phase estimation. Still, the observed trade-off is relatively small and shrinks with increasing  $N$ . We conjecture that for asymptotically large  $N$  the discrepancy is vanishing and the simultaneous scheme performs as well as the separate one. This is presented in Fig. 4 where the average of estimation uncertainties of  $\varphi$  and  $\eta$  achievable when utilizing two-axis spin-squeezed states [44] normalized according to the

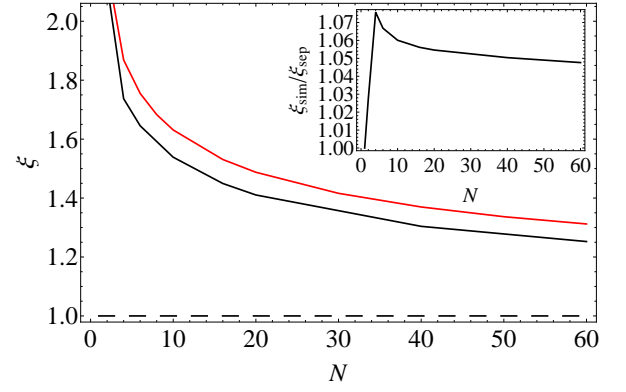


FIG. 4. Average  $\xi$  of normalized uncertainties of estimating the phase and the dephasing parameter in the optimal simultaneous scheme (solid, red) and the optimal separate scheme (solid, black) as a function of the number of atoms used and the dephasing parameter set to  $\eta = 0.9$ . For the separate scheme the considered average asymptotically saturates to 1, which is represented by black dashed line. The inset indicates the ratio of the precision achieved in both schemes indicating that the discrepancy is relatively small and decreases with increasing  $N$  which indicates the possibility of satisfying the compatibility requirement in the asymptotic regime of large  $N$ .

asymptotic optimal performance of the separate scheme

$$\xi = \frac{1}{2} \left( \frac{\Delta^2\varphi}{(1-\eta^2)/(\eta^2 N)} + \frac{\Delta^2\eta}{(1-\eta^2)/N} \right) \quad (50)$$

is plotted. When the above quantity is calculated for separate and simultaneous schemes for  $\eta = 0.9$  the resulting discrepancy is maximal for  $N = 4$  when it achieves about 7.6% and decreases with increasing  $N$  going below 4.8% for  $N = 60$  (see the inset). For smaller  $\eta$  the discrepancy is even smaller although its maximum is attained for larger  $N$ . This numerics strongly suggests that asymptotically simultaneous scheme can perform as well as the separate one. The two-axis spin squeezed states used as an input probe here are parameterized using the squeezing parameter  $\theta$  as  $|\psi_\theta\rangle = e^{-i\theta(J_+^2 - J_-^2)}|j,j\rangle$  where  $J_+$ ,  $J_-$  are standard angular momentum ladder operators and the dependence of optimal squeezing parameter as a function of  $N$  is approximately  $\theta \sim N^{-0.9}$  when  $\eta = 0.9$ . We have also checked the behavior of one-axis spin-squeezed states, recently used in quantum enhanced magnetometry [45, 46], which are defined as  $|\psi_\theta\rangle = U_\phi^{SSS} e^{-i\theta J_x^2}|j,j\rangle$ , where  $J_x$  is  $x$  component of the angular momentum,  $U_\phi^{SSS} = e^{iH\phi}$  denotes unitary transformation generated by operator  $H = e^{i\theta J_x^2} J_z e^{-i\theta J_x^2}$  and  $\phi = \frac{1}{4} \arctan \left[ \frac{4 \sin \theta (\cos \theta)^{N-2}}{1 - [\cos(2\theta)]^{N-2}} \right]$ . Surprisingly, we have found that such states give significantly worse results and do not allow to saturate the performance of the separate scheme.

These conclusions are therefore similar to the ones obtained in [20] where a different model assuming collec-

tive instead of uncorrelated dephasing was analyzed and again asymptotic possibility of performing optimal simultaneous estimation of phase and the dephasing parameter has been demonstrated.

For clarity, we comment on the applicability of the above conclusions in the context of Fig. 2. Although we are considering both entangled states and measurements [43], the scheme can be considered in the model of Fig. 2 (b), where now each of  $\rho$  is a large entangled state upon which we perform an even larger entangled measurement and thus the results of Section IV apply without complication.

## VII. CONCLUSIONS

We have presented a complete analysis of the compatibility problem in multiparameter quantum metrology, pointing out three main obstacles to estimating parameters simultaneously with the same accuracy as in the separate scenario. We have provided several examples which illustrate how these obstructions come into force, as well as being interesting in their own right.

We would like to stress, however, that multiparameter metrology is not all about trying to avoid an overwhelming array of pitfalls. In this paper we have taken the specific approach in which we were asking for a multiparameter protocol to meet the performance of the separate scheme where each of the parameters is estimated independently with the highest possible precision possible. Clearly, even if a multiparameter scheme cannot meet this condition, it does not mean that there is no advantage in estimating multiple parameters simultaneously. In a  $p$ -parameter estimation problem, the separate scheme that we have used as a reference consumes  $p$  times as much resources as the simultaneous scheme. There-

fore, in general there will be an advantage coming from simultaneous estimation even if the compatibility conditions are not satisfied. This has indeed been the line of research of many other papers dealing with multiparameter metrology. From this point of view, one can view this paper as providing a systematic view on the situation when multiparameter estimation manifests its maximal advantage over separate schemes by meeting their performance while consuming a factor of  $p$  fewer resources.

It is also interesting to comment on the issue of sequential vs. parallel schemes in quantum metrology in the multiparameter case. It is known that in decoherence-free single unitary parameter estimation a scheme where unitaries act sequentially on a single probe provides the same maximal QFI as the parallel scheme where one allows arbitrary input entangled state of  $N$  particles to be sent through  $N$  parallel unitaries [31] and only the presence of decoherence makes the schemes inequivalent [47]. We have shown that using two-qubit entangled input states allows one to optimally estimate two rotation angles around perpendicular axes with precision equal to that which could be obtained in the separate scheme. Clearly, this could not be achieved by acting sequentially with two unitaries on a single qubit as in this case we have proven that the compatibility condition cannot be satisfied when two parameters are to be estimated. This breaks the equivalence between entangled and sequential unitary parameter estimation in the multiparameter case.

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- [1] A. S. Holevo, *Probabilistic and statistical aspects of quantum theory* (North-Holland, 1982).
  - [2] C. W. Helstrom, *Journal of Statistical Physics* **1**, 231 (1969).
  - [3] V. Giovannetti, S. Lloyd, and L. Maccone, *Nature Photonics* **5**, 222 (2011).
  - [4] S. L. Braunstein and C. M. Caves, *Physical Review Letters* **72**, 3439 (1994).
  - [5] G. Tóth and I. Apellaniz, *Journal of Physics A: Mathematical and Theoretical* **47**, 424006 (2014).
  - [6] R. Demkowicz-Dobrzański, M. Jarzyna, and J. Kołodyński, in *Progress in Optics*, Vol. 60, edited by E. Wolf (Elsevier, 2015) pp. 345–435, arXiv:1405.7703 [quant-ph].
  - [7] K. Matsumoto, *Journal of Physics A: Mathematical and General* **35**, 3111 (2002).
  - [8] M. Hayashi, ed., *Asymptotic theory of quantum statistical inference*, Vol. 1 (World Scientific, 2005).
  - [9] R. D. Gill and M. Guta, *ArXiv e-prints* (2011), arXiv:1112.2078 [quant-ph].
  - [10] E. Bagan, M. Baig, and R. Muñoz Tapia, *Phys. Rev. Lett.* **87**, 257903 (2001).
  - [11] G. Chiribella, G. M. D'Ariano, P. Perinotti, and M. F. Sacchi, *Phys. Rev. Lett.* **93**, 180503 (2004).
  - [12] P. Kolenderski and R. Demkowicz-Dobrzanski, *Phys. Rev. A* **78**, 052333 (2008).
  - [13] C. Vaneph, T. Tufarelli, and M. G. Genoni, *Quantum Measurements and Quantum Metrology* **1**, 12 (2013).
  - [14] M. A. Ballester, *Physical Review A* **69**, 022303 (2004).
  - [15] O. Barndorff-Nielsen and R. Gill, *Journal of Physics A: Mathematical and General* **33**, 4481 (2000).
  - [16] A. Fujiwara, *Physical Review A* **65**, 012316 (2001).
  - [17] P. C. Humphreys, M. Barbieri, A. Datta, and I. A. Walmsley, *Physical review letters* **111**, 070403 (2013).
  - [18] D. W. Berry, M. Tsang, M. J. W. Hall, and H. M. Wiseman, *Phys. Rev. X* **5**, 031018 (2015).
  - [19] T. Baumgratz and A. Datta, *Phys. Rev. Lett.* **116**, 030801 (2016).

- [20] S. I. Knysh and G. A. Durkin, ArXiv e-prints (2013), arXiv:1307.0470 [quant-ph].
- [21] M. D. Vidrighin, G. Donati, M. G. Genoni, X.-M. Jin, W. S. Kolthammer, M. Kim, A. Datta, M. Barbieri, and I. A. Walmsley, *Nature communications* **5** (2014).
- [22] P. J. Crowley, A. Datta, M. Barbieri, and I. A. Walmsley, *Physical Review A* **89**, 023845 (2014).
- [23] A. Monras and F. Illuminati, *Physical Review A* **83**, 012315 (2011).
- [24] M. Szczykulska, T. Baumgratz, and A. Datta, ArXiv e-prints (2016), arXiv:1604.02615 [quant-ph].
- [25] S. M. Kay, *Fundamentals of statistical signal processing: estimation theory* (Prentice Hall, 1993).
- [26] D. R. Cox and N. Reid, *Journal of the Royal Statistical Society. Series B (Methodological)*, 1 (1987).
- [27] H. Nagaoka, in *Asymptotic theory of quantum statistical inference*, Vol. 1, edited by M. Hayashi (World Scientific, 1989) Chap. 8.
- [28] M. Hayashi and K. Matsumoto, *Journal of Mathematical Physics* **49**, 102101 (2008).
- [29] J. Kahn and M. Guță, *Communications in Mathematical Physics* **289**, 597 (2009).
- [30] K. Yamagata, A. Fujiwara, R. D. Gill, *et al.*, *The Annals of Statistics* **41**, 2197 (2013).
- [31] V. Giovannetti, S. Lloyd, and L. Maccone, *Physical review letters* **96**, 010401 (2006).
- [32] R. Demkowicz-Dobrzanski, U. Dorner, B. Smith, J. Lundeen, W. Wasilewski, K. Banaszek, and I. Walmsley, *Physical Review A* **80**, 013825 (2009).
- [33] M. Jarzyna and R. Demkowicz-Dobrzański, *Phys. Rev. Lett.* **110**, 240405 (2013).
- [34] S. Knysh, V. N. Smelyanskiy, and G. A. Durkin, *Phys. Rev. A* **83**, 021804 (2011).
- [35] J. Kolodyński and R. Demkowicz-Dobrzanski, *Phys. Rev. A* **82**, 053804 (2010).
- [36] B. M. Escher, R. L. de Matos Filho, and L. Davidovich, *Nature Phys.* **7**, 406 (2011).
- [37] R. Demkowicz-Dobrzański, J. Kołodyński, and M. Guță, *Nature communications* **3**, 1063 (2012).
- [38] E. Tesio, S. Olivares, and M. G. Paris, *International Journal of Quantum Information* **9**, 379 (2011).
- [39] S. F. Huelga, C. Macchiavello, T. Pellizzari, A. K. Ekert, M. Plenio, and J. Cirac, *Physical Review Letters* **79**, 3865 (1997).
- [40] D. Ulam-Orgikh and M. Kitagawa, *Phys. Rev. A* **64**, 052106 (2001).
- [41] A. Fujiwara and H. Imai, *J. Phys. A: Math. Gen.* **36**, 8093 (2003).
- [42] J. Kolodyński and R. Demkowicz-Dobrzanski, *New Journal of Physics* **15**, 073043 (2013).
- [43] M. Jarzyna and R. Demkowicz-Dobrzański, *New Journal of Physics* **17**, 013010 (2015).
- [44] J. Ma, X. Wang, C. Sun, and F. Nori, *Physics Reports* **509**, 89 (2011).
- [45] W. Muessel, H. Strobel, D. Linnemann, D. B. Hume, and M. K. Oberthaler, *Phys. Rev. Lett.* **113**, 103004 (2014).
- [46] R. J. Sewell, M. Koschorreck, M. Napolitano, B. Dubost, N. Behbood, and M. W. Mitchell, *Phys. Rev. Lett.* **109**, 253605 (2012).
- [47] R. Demkowicz-Dobrzanski and L. Maccone, ArXiv e-prints (2014), arXiv:1407.2934 [quant-ph].